

## ON THE PROBLEM OF HOMOGRAPHIC SOLUTIONS OF THE THREE-BODY PROBLEM

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This paper deals with the problem of the existence of homographic solutions of the three-body problem in the case of the law of the inverse cube of the mutual distances. It is discussed a solution given by Dr. R.P. Cesco in his paper Sobre las soluciones homográficas del problema de los tres cuerpos, Pub. Observ. Astron. La Plata, Serie Astronómica, Tomo XXV, N 2, 1959. We quote from this paper: "The object of this paper is to prove the following theorem: The only homographic solutions of the three-body problem of celestial mechanics with a law of attraction proportional to any power  $r^\alpha$  of the distance  $r$  are: (I) The pure dilatations. (II) The collinear solutions. (III) The equilateral solutions. (IV) The isosceles solutions of Banachiewicz for  $\alpha = 3$ , and (V) The scalene solutions given in this note, also for  $\alpha = 3$ , the first three kinds being the only planar solutions for any value of  $\alpha$ ".

We shall show now that: 1st, Dr. Cesco's vectorial definition of the homographic solutions is such that the vectors used in the statement do not transform as tensors. This means that the vector products can not be equivalent to "rotational" matrices in the Euclidean space to represent rotations. In other words, this is equivalent to say that the vectors used by Dr. Cesco to define an homographic solution, as being of constant modulus and functions of the time, cannot represent a set of "continuous" infinitesimal rotations. Thus they do admit only one interpretation: they are vectors defined in the ordinary sense of the analytical geometry. 2nd. It is shown as a consequence that Dr. Cesco's "vectorial" solution  $\vec{P} = \lambda \vec{Q} + \vec{R}$  where  $\vec{P}$  and  $\vec{Q}$  are "vectors" of constant modulus and functions of the time,  $\vec{R}$  is a constant vector and  $\lambda$  is a positive scalar, does not represent any kind of homographic solutions under the assumption made to solve the problem.

1.- The problem of homographic solutions has been extensively treated especially during the present century. The problem can be tackled from different points of view. It has been shown that the vectorial treatment is possible if the vectors of the statement transform as tensors. In this way vector products can be equivalent, in regard to rotations, to the orthogonal matrices introduced in the classical statement of the problem. It is very well known that "ordinary" vectors, i.e., those defined in the sense of the analytical geometry cannot represent finite rotations. But vectors do represent infinitesimal rotations. In order to get this representation, vectors must transform as tensors. In this way finite rotations represented by vector products will be equivalent to a "continuous" set of small rotations.

2.- The classical definition of homographic solution states that there exists such a solution when the configuration of the  $n$  bodies remains similar to itself for every of the time  $t$ .

This means that there exist an orthogonal matrix  $\Omega$  function of the time, a scalar  $r(t)$ , the dilatation such that the configuration at time  $t$  is given by  $\xi = r \Omega \xi^0$  (1) where  $\xi$  is the position vector at time  $t$ , and  $\xi^0$  is the position vector for some initial time  $t^0$ . It is to be understood that  $\xi^0$  is a constant vector.

Definition (1) implies that a barycentric inertial system of reference has been set up.

We can also define the homographic solutions in terms of non inertial barycentric system of coordinates, i.e. a rotating set of coordinates, in the following way

(2)  $x = r \xi^0$  where  $x = \Omega^{-1} \xi$  indicates a rotating set of reference. We draw attention that the vectors thus defined must transform as tensors.

3.- We observe now that Dr. Cesco's definition of an homographic solution has been set up with respect to an "heliocentric" system of reference. In fact he writes: (3)  $r_1 = \rho P$   $r_2 = \rho Q$ ; where  $P$  and  $Q$  are vectors of constant modulus and functions of the time;  $\rho$  is the dilatation;  $r_1$  and  $r_2$  are the position "vectors" of the masses  $m_1$  and  $m_2$  respect to the mass  $m_0$  at center of which the origin of the system of reference is placed. It is clear that "heliocentric" systems of reference are not inertial in general. Now it is also clear that from Dr. Cesco's formulae (4)  $P^2 = c_1$   $Q^2 = c_2$  where  $c_1$  and  $c_2$  are constants. We cannot suppose that "vectors"  $P$  and  $Q$  admit the following definitions (5)  $P = \Omega(t) P^0$   $Q = \Omega(t) Q^0$ , where  $\Omega(t)$  is an orthogonal matrix function of the time,  $P^0$  and  $Q^0$  would be the values of  $P$  and  $Q$  at some fixed  $t^0$ , respectively. If it were the case, it should occur that the elements of that matrix  $\Omega(t)$  should be constant against the hypothesis that these elements are functions of the time. Of course there is no rotation at all in that case, that is no homographic solution should be possible. Besides, that the vectors  $P$  and  $Q$  are defined in the "ordinary" sense it is seen from the examples given by Dr. Cesco at the end of his paper. We must also point out that no rotating system of reference in the sense of formulae (2) has been introduced. All this shows that Dr. Cesco's vectors  $P$  and  $Q$  do not transform as tensors. So they do not have the same meaning as the vectors defined by formulae (2). There is indeed only one interpretation in view of the aforesaid proofs. We emphasize the fact that under the assumption made by Dr. Cesco it arises only one interpretation: that the vectors  $P$  and  $Q$  functions of the time and of constant modulus are used to represent "really" finite rotations.

In order to verify this interpretation we only need to study the behaviour of Dr. Cesco's "vectorial" solution: (6)  $\vec{P} = \lambda \vec{Q} + \vec{R}$  where  $\vec{R}$  is a constant vector and  $\lambda$  is a scalar such that  $\lambda > 0$  (the case  $\lambda = 0$  must be excluded because theruse, we will have a division by zero).

Following a proof given in Goldstein's Analytical Mechanics it can be easily shown that for the "ordinary" vectors defined in Dr. Cesco's solution the commutative law respect to the sum does not hold i.e. it follows immediatly from (6) that (7)  $\vec{P} + \vec{Q} \neq \vec{Q} + \vec{P}$

From this we deduce that the magnitudes  $P$  and  $Q$  cannot be accepted as vectors. From this too, it results that no homographic solution can be obtained from the use of the equation

$$\vec{P} = \lambda \vec{Q} + \vec{R}.$$

In this way we have completed the discussion and verified the statement made at the beginning.